Mock Mandelbrot #1 Solutions

MVMSC

9/29/2017

1. What is the largest possible area of a triangle with side lengths 28 and 45?
   The area of a triangle is given by $\frac{1}{2}ab\sin\theta$ where a, b are two side lengths and θ is the angle between them. This expression is maximized when $\sin\theta = 1$, or $\theta = 90°$. In this case, the maximum area $\frac{1}{2}(28)(45)(1) = \boxed{630}$ is achieved with a right triangle with legs 28 and 45.

2. Find the largest integer value of x such that $\frac{2x^2+52x+120}{x+26}$ is an integer.
   Divide the numerator by the denominator, so the expression is equal to $2x + \frac{120}{x+26}$. For this to be an integer, $x + 26 \mid 120$. To maximize x, maximize $x + 26$ as 120 (as something larger clearly cannot divide 120) so the answer is $120 - 26 = \boxed{94}$.

3. How many zeroes are at the end of $1^1 \times 2^2 \times \cdots \times 30^{30}$?
   To count the number of zeroes, find the greatest power of 10. Clearly, there are more factors of 2 than 5, so we will count the power of 5. We only need to consider terms that have bases that are multiples of 5, so we have in $5^5 10^{10} 15^{15} 20^{20} 25^{25} 30^{30}$ a power of $5^{5+10+15+20+2(25)+30} = 5^{130}$. Therefore there are $\boxed{130}$ zeroes at the end of the number.

4. 8 distinct equilateral triangles with side length 12 are constructed on the side of a square with side length 12 so that each triangle shares a side with the square. What is the area of the region inside the quadrilateral formed by the 4 triangle centers outside the square but not the quadrilateral formed by the 4 triangle centers inside the square?
   Note that connecting each set of four centers results in a square. To find the square, one can find the diagonal d, and the square will have area $\frac{d^2}{2}$. Consider the distance from the center of a triangle to a side, and let it be x. Then the diagonals for the squares, respectively, are $12 + 2x$ and $12 - 2x$, so the difference between the areas is $\frac{(12+2x)^2-(12-2x)^2}{2} = \frac{(4x)(24)}{2} = 48x$. To find x, drop a perpendicular to the edge and draw lines to a vertex of the edge. This forms a $30 - 60 - 90$ triangle with x opposite the $30°$ angle and $\frac{12}{2} = 6$ opposite the $60°$ angle, so $x = \frac{6}{\sqrt{3}} = 2\sqrt{3}$. Evaluating for the area, $48x = \boxed{96\sqrt{3}}$.

5. Alice randomly selects a positive integer between 1 and 2047, inclusive. Bob guesses Alice's integer, to which Alice replies if her integer is larger, smaller, or equal to Bob's guess. What is the minimum number of guesses needed so that Bob is guaranteed to know Alice's number?
   In general, given a range from 1 to $2^n - 1$, Bob needs $n - 1$ guesses to find Alice's number. As a base case, if $n = 2$, Bob can guess 2, after which he will know Alice's number in one guess (if she says lower, it is 1, and if higher, it is 3). Now, assume that for a range of length $2^{n-1} - 1$, Bob can achieve $n - 2$ guesses. For a range 1 to $2^n - 1$, Bob can guess 2^{n-1}, reducing the problem to one of two segments of length $2^{n-1} - 1$ which each need $n - 2$ guesses. Then he can find Alice's number in $n - 2 + 1 = n - 1$ guesses. By induction, Bob needs $n - 1$ guesses for 1 to $2^n - 1$. Then the answer to the problem is $\boxed{10}$ since $2047 = 2^{11} - 1$.

6. Find the last three digits of the closest integer to $(\sqrt{13} + \sqrt{10})^{804}$.
   To solve this problem, we will instead find the value of $(\sqrt{13}+\sqrt{10})^{804}+(\sqrt{13}-\sqrt{10})^{804}$. Using the binomial theorem, this equals $\sum_{i=0}^{804} \binom{804}{i} \sqrt{13}^{804-i} \sqrt{10}^{i} + \sum_{i=0}^{804} \binom{804}{i} \sqrt{13}^{804-i} (-\sqrt{10})^{i} =$ $\sum_{i=0}^{804} \binom{804}{i} \sqrt{13}^{804-i} \sqrt{10}^{i} (1+(-1)^i)$. Note that each term in this sum with odd index i evaluates to zero, and each term with even i is an integer because the square roots are raised to even powers. This means that this sum is an integer, and in fact, this value is equal to the closest integer to $(\sqrt{13} + \sqrt{10})^{804}$, as the second term is a small decimal much less than 0.5. Now, note that we are only considering even i, and because we want the last three digits, we do not need to consider any term with $i \geq 6$ because $\sqrt{10}^6 = 1000$ so any such term would contribute zero to the answer. Therefore we only need to evaluate the expression $2\left(\binom{804}{4}\sqrt{13}^{800}\sqrt{10}^{4} + \binom{804}{2}\sqrt{13}^{802}\sqrt{10}^{2} + \binom{804}{0}\sqrt{13}^{804}\sqrt{10}^{0}\right) =$ $2\left(\binom{804}{4}13^{400}10^2 + \binom{804}{2}13^{401}10^1 + \binom{804}{0}13^{402}10^0\right) \pmod{1000}$. One can compute that $\binom{804}{4} \equiv \binom{804}{0} \equiv 1 \pmod{1000}$ and $\binom{804}{2} \equiv 806 \pmod{1000}$. In addition, by Euler's theorem, $13^{\phi(1000)} = 13^{400} \equiv 1 \pmod{1000}$. Therefore our expression equals $2((1)(1)(100) + (806)(13)(10) + (1)(169)(1)) \equiv \boxed{098} \pmod{1000}$.

7. Eric has red, blue and, yellow paint. In how many ways, distinct up to rotation, can he color a cube?
   There are two solutions, casework and Burnside's lemma.

   (a) Casework on the number of sides by the majority color.

   - If the number of sides is 6, there are 3 ways, one for each color.
   - If the number of sides is 5, there are $3 \cdot 2 = 6$ ways by choosing colors, since there is only one arrangement up to rotation.
   - If the number of sides is 4, then the two sides that are not this majority color may be adjacent or opposite. For each case, there are $3 \cdot 3 = 9$ ways; 3 to choose the majority color and 3 to choose the colors of the two other sides, which may be the same (for each of two colors) or different. This gives a total of 18 for this case.

- If the number of sides is 3, note that for a fixed color, two sides must be adjacent. There are two cases for the third side, if it is adjacent to both of these sides or only one. If it is adjacent to both, consider the three other sides. If they are all of the same color, there are $\frac{3\cdot 2}{2} = 3$ ways, adjusting for overcounting. If there are two of one color and one of the other, there are $3 \cdot 2 \cdot 1 = 6$ ways by choosing the colors. If the third side of the majority color is adjacent to only one side of the same color, consider the colors of the other three sides. If they are all the same, there are again 3 ways. If there are two of one color and one of the other, there are again 6 ways to choose the colors, and 2 ways to arrange the two of the color (whether they are adjacent or not) for a total of 12 ways. Thus this case totals $3 + 6 + 3 + 12 = 24$ ways.
- If the number of sides is 2, then each color must be on two sides. Consider for a fixed color two cases, whether they are adjacent or opposite. If they are opposite, there are 2 ways to arrange the other two pairs up to rotation, again opposite or adjacent. If they are adjacent, consider the colors of the two sides that each border these two sides of fixed color. There are 2 ways they may be the same, and if they are different, there are 2 ways to arrange the other sides for a total of 6 in this case.

Therefore there are $3 + 6 + 18 + 24 + 6 = \boxed{57}$ ways to color the cube.

(b) Burnside's lemma is a theorem in group theory of which the formal definition can be found online. Its combinatorial interpretation provides a method to count the number of colorings by considering symmetries, or rotations that would leave the colors apparently unchanged. For each such rotation, add the number of colorings that would be kept the same by applying that rotation, and divide this sum by the total number of rotations. Specifically, there are:

- one identity rotation, or leaving the cube unchanged. All 3^6 colorings are unchanged by this, so add this to the sum.
- six $90°$ face rotations, each of which leaves 3^3 colorings unchanged, as the four colors around the axis of rotation must be the same for the rotation to not change the coloring.
- three $180°$ face rotations, each of which leaves 3^4 colorings unchanged, as each opposite pair of sides around the axis of rotation must share colors
- eight $120°$ vertex rotations, each of which leaves 3^2 colorings unchanged, as each face bordering this vertex must have the same color, and the same for the opposite vertex.
- six $180°$ edge rotations, each of which leaves 3^3 colorings unchanged. This is a rotation about the line connecting the midpoints of two opposite edges, so three pairs of faces switch location and therefore must share the same color.

Therefore the answer is $\frac{1}{24}(3^6 + 6 \cdot 3^3 + 3 \cdot 3^4 + 8 \cdot 3^2 + 6 \cdot 3^3) = \boxed{57}$ colorings.