Mock Mandelbrot #2 Solutions

MVMSC

11/3/2017

1. What is the sum of the squares of the first 10 positive integers?
   In general, the sum $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, which can be proved by induction. Plugging in $n = 10$, the sum equals $\frac{10\cdot 11\cdot 21}{6} = \boxed{385}$.

2. Find the radius of a circle with area equal to its circumference.
   Since area equals πr^2 and circumference equals $2\pi r$, $\pi r^2 = 2\pi r$ implies that the radius $r = \boxed{2}$.

3. Let r and s be the roots of the quadratic $x^2 - 16x + 64 = 0$. Find $\frac{1}{r} + \frac{1}{s}$.
   Combining the fractions, $\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs}$. By Vieta's Formulas, $r + s = -\frac{-16}{1} = 16$ and $rs = \frac{64}{1} = 64$, so the desired quantity equals $\frac{16}{64} = \boxed{\frac{1}{4}}$

4. How many distinct right triangles with integer side lengths have a hypotenuse of length 65?
   It is well known that all primitive Pythagorean triples can be expressed in the form $(m^2 - n^2, 2mn, m^2 + n^2)$ for positive integers $m > n > 0$ such that $\gcd(m, n) = 1$ and m and n are not both odd (this fact is known as **Euclid's Formula**). For a hypotenuse of 65, the triangle must either be a primitive triple or a multiple of a primitive triple with hypotenuse 5 or 13 (it is impossible to have a hypotenuse of 1). We now split into cases based on this value.

   - If $m^2 + n^2 = 5$, then $m = 2, n = 1$, and the primitive Pythagorean triple is $(3, 4, 5)$. Scaling by a factor of 13, one triangle is $(39, 52, 65)$.
   - If $m^2 + n^2 = 13$, then $m = 3, n = 2$, and the primitive Pythagorean triple is $(5, 12, 13)$. Scaling by a factor of 5 one triangle is $(25, 60, 65)$.
   - If $m^2 + n^2 = 65$, then $m = 7, n = 4$ or $m = 8, n = 1$, and the primitive Pythagorean triples, which do not need to be scaled, are $(33, 56, 65)$ and $(16, 63, 65)$, respectively.

   Therefore there are $\boxed{4}$ such triangles.

5. How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?
Let the degree measures be $a-b, a, a+b$ for integer degree measures a, b. Then their sum $(a-b)+a+(a+b) = 3a = 180°$, so $a = 60°$. Then, since $a - b > 0$ and $b > 0$, $0° < b < 60°$, a total of $\boxed{59}$ integer possibilities, each of which will generate a non-similar triangle.

6. How many pairs of positive integers (m, n), with m and n both less than 100, have the property that $m^3 - n^3 = 5$?
Factoring difference of cubes, $(m-n)(m^2+mn+n^2) = 5$. Since $m^2+mn+n^2 \geq 3$, we must have that $m - n = 1$, and therefore $m^2+mn+n^2 = 3n^2+3n+1 = 5$, which yields no integer solutions. Therefore there are $\boxed{0}$ such pairs.

7. Let ABC be a triangle with $AB = 20$, $BC = 27$, and $\sin^2 A + \sin A \sin C + \sin^2 C = \sin^2 B$. What is the area of ABC?
By the Law of Sines, $\frac{\sin C}{20} = \frac{\sin A}{27} = \frac{\sin B}{AC}$. Therefore $\sin^2 A + \sin A \sin C + \sin^2 C = \frac{\sin^2 B}{AC^2} \cdot 27^2 + \frac{\sin B}{AC} \cdot 27 \cdot \frac{\sin B}{AC} \cdot 20 + \frac{\sin^2 B}{AC^2} \cdot 20^2 = \sin^2 B$. Dividing by $\sin^2 B$ and multiplying by AC^2, we have that $AC^2 = 20^2 + 27^2 + 20 \cdot 27$. By the Law of Cosines, $AC^2 = 20^2 + 27^2 + 20 \cdot 27 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cos B = 20^2 + 27^2 - 2 \cdot 20 \cdot 27 \cos B$, so $\cos B = -\frac{1}{2}$. Then $\sin B = \frac{\sqrt{3}}{2}$, and the area of the triangle equals $\frac{1}{2}(AB)(BC)\sin B = \frac{1}{2}(20)(27)\frac{\sqrt{3}}{2} = \boxed{135\sqrt{3}}$.