Mock Mandelbrot #3 Solutions

MVMSC

1/19/2018

1. What is the largest positive integer that can't be expressed as the sum of positive multiples of 5 and 7?
   Note that once we construct a solution with a certain remainder $\pmod 5$, we can construct all greater numbers that share that remainder. It remains to find the smallest number we can construct with each remainder $\pmod 5$. Since adding 5 doesn't change the remainder, we only need to consider multiples of 7, so that $0 \equiv 0, 7 \equiv 2, 14 \equiv 4, 21 \equiv 1, 28 \equiv 3 \pmod 5$ and the largest number that we cannot express is $28 - 5 = \boxed{23}$. (In general, the answer is $ab - a - b$. This is the $n = 2$ case of the Frobenius Coin Problem.)

2. How many numbers between 1000 and 9999 have the property that their digits are in strictly increasing order?
   Note that given any 4 digits from 1 to 9, there is a unique number that satisfies the conditions. Therefore the answer is $\binom{9}{4} = \boxed{126}$.

3. A positive integer has 9 positive factors. What is the sum of all possible numbers of factors its cube can have?
   If a number $n = \prod_{p_i | n} p_i^{e_i}$, then the number of factors is $\prod_{p_i | n} (e_i + 1)$. Since $9 = 9 = 3 \cdot 3$, there are two possible cases, either $n = p^8$ or $n = p^2 q^2$ for prime p, q. Then n^3 equals $p^2 4$ and $p^6 q^6$ respectively and the sum of the number of factors is $(24+1)+(6+1)(6+1) = 25 + 49 = \boxed{74}$.

4. A solid cube has side length 9 inches. A 8-inch by 8-inch square hole is cut into the center of each face. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume, in cubic inches, of the remaining solid?
   These cuts leave behind a frame where each "edge" of the remaining solid is a $\frac{1}{2} \times \frac{1}{2} \times 9$ rectangular prism. Each $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$ corner is overcounted twice, so the answer is $\frac{9}{4} \cdot 12 - 2(\frac{1}{8} \cdot 8) = 27 - 2 = \boxed{25}$.

5. Triangle ABC has $AC = 75$ and $BC = 50$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 30$, find LP.
   We proceed using mass points. By the Angle Bisector Theorem, we may assign B a mass of 3 and A a mass of 2 so the mass of L is 5. Then, since K is the midpoint of AC, the mass of C is also 2. Therefore $\frac{LP}{PC} = \frac{2}{5}$. Since K is the midpoint of AC, $AK = KC$, and since K is the midpoint of PM, $PK = KM$. The angles PKC and AKM are vertical angles and equal, so therefore $\triangle PKC \cong \triangle MKA$ and $PC = AM = 30$. Therefore $LP = \frac{2}{5}PC = \frac{2}{5}(30) = \boxed{12}$.

6. Suppose that x, y, and z are three positive numbers that satisfy the equations $xyz = 1$, $x + \frac{1}{z} = 13$, and $y + \frac{1}{x} = 17$. Find $z + \frac{1}{y}$.
   For the equation $x + \frac{1}{z} = 13$, combine with a common denominator and multiply by $\frac{xy}{xy}$ so that $\frac{xz(xy)+xy}{xyz} = \frac{x+xy}{1} = x + xy = 13$, and multiply by z on both sides so $xz + 1 = 13z$. Similarly, the other equation gives $y + yz = 17$ and $xy + 1 = 17x$. Substitute $xy = 17x - 1$ into $x + xy = 13$ to find that $x = \frac{7}{9}$, and solve so that $y = \frac{9}{110}$ and $z = \frac{110}{7}$. Therefore $z + \frac{1}{y} = \frac{9}{110} + \frac{7}{110} = \boxed{\dfrac{8}{55}}$.

7. Triangle ABC has side lengths $AB = 5, BC = 7, CA = 6$. Let ω be the circumcircle of ABC and I be the incenter of ABC. Line AI intersects ω again at point X. Find the length of IX.
   Lemma: $XC = XB = XI$ (Fact 5).
   Proof: Angle $XAC = XBC$ since they intercept the same arc. Since AI and BI are angle bisectors, angles $XAC = XAB$ and $IBA = IBC$. Then angle $XIB = XAB + IBA = XAC + IBC = XBC + IBC = XBI$, so $\triangle XBI$ is isosceles with $XB = XI$. Similarly, $XC = XI$, so $XC = XB = XI$. $\square$
   Now, the Law of Cosines on $\triangle ABC$ gives $7^2 = 6^2 + 5^2 - 2 \cdot 6 \cdot 5 \cdot \cos(BAC)$ and $\cos(BAC) = \frac{1}{5}$. Since $ABXC$ is a cyclic quadrilateral, angle $BAC + BXC = \pi$ and $\cos(BXC) = -\cos(BAC) = -\frac{1}{5}$. Let $XC = XB = x$, and Law of Cosines on triangle BXC gives $7^2 = x^2 + x^2 - 2x^2\cos(BXC) = \frac{12}{5}x^2$ and $x = \boxed{\dfrac{7\sqrt{15}}{6}}$.