Mock Mandelbrot #4 Solutions

MVMSC

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Problems and solutions from *American Regions Math League & ARML Power Contests 1995-2003* by Donald Barry and Thomas Kilkelly.

Test arranged by Ethan Guo

1. A pro athlete played for 17 years and earned 72 million dollars. She was paid k million a year where k is an integer and received an extra one million each year that her team made the playoffs. Compute the number of years her team made the playoffs. (1999 ARML I-1)
   $17k + n = 72 \to k = \frac{72-n}{17}$ for $0 \le n \le 17$. The only solution is $n = \boxed{4}$.

2. Compute the largest prime factor of
   $$3(3(3(3(3(3(3(3(3(3(3+1)+1)+1)+1)+1)+1)+1)+1)+1)+1)+1$$
   (1995 ARML I-1)
   The given expression equals $3^{11} + 3^{10} + 3^9 + \ldots + 3^1 + 1 = \frac{3^{12}-1}{3-1}$. The numerator factors as $(3^6+1)(3^6-1) = (3^2+1)(3^4-3^2+1)(3^3-1)(3^3+1) = 10 \cdot 73 \cdot 26 \cdot 28$. The largest prime factor is $\boxed{73}$.

3. The sides of a non-right isosceles $\triangle ABC$ are $\sin x$, $\cos x$, and $\tan x$. Compute $\sin x$. (1998 ARML I-4)
   If $\sin x = \cos x$, then $x = 45^\circ$, the third side $= \tan 45^\circ = 1$, but the sides are $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, and 1, making a right triangle. If $\sin x = \tan x$, then $\sin x = 0$ or $\cos x = 1$ which makes $\sin x = 0$ and no triangle can be formed. Thus, $\cos x = \tan x \to \cos^2 x = \sin x \to 1 - \sin^2 x = \sin x \to \sin^2 x + \sin x - 1 = 0 \to \sin x = \boxed{\dfrac{-1+\sqrt{5}}{2}}$.

4. There are n triangles of positive area that have one vertex at $A(0,0)$ and the other two vertices at points with coordinates in $\{0,1,2\}$. Compute n. (2002 ARML TB-3)
   There are $3 \cdot 3 = 9$ points that can be formed using the coordinates $\{0,1,2\}$. Eliminate $A(0,0)$ and from the remaining 8 points choose 2. This can be done in ${}_8C_2 = 28$ ways. Eliminate the pairs $(0,1)$ and $(0,2)$, $(1,1)$ and $(2,2)$, $(1,0)$ and $(2,0)$ since they are collinear with A, leaving $28 - 3 = \boxed{25}$ pairs of points.

5. If, from left to right, the last seven digits of $n!$ are 8000000, compute the value of n. (2000 ARML I-5)
Since $n!$ ends in 6 zeroes, $n!$ must have exactly 6 fives in its prime factorization, so we know that $25 \leq n \leq 29$. Consider the units digit of $\frac{n!}{10^6}$. For $n = 25$, a quick multiplication by all remaining factors yields a units digit of 4. For $n = 26$, the units digit is also 4, but for $n = 27$ it is 8. Thus, $n = \boxed{27}$.
Note that for $n = 28$, the units digit is 4 and for $n = 29$, it is 6.

6. Let circles O and Q have a common chord $\overline{PS}$. If $OQ = 324$ and $MN : NT = 2 : 1$, compute $OP - PQ$. (The intersections of segment $\overline{OQ}$ with circle Q, segment $\overline{PS}$, and circle O are M, N, T respectively) (1998 ARML TB-1)
Let the radius of circle O be R and the radius of circle Q be r. Let $MN = 2x$ and $NT = x$. Then $ON = R - x$ and $QN = r - 2x$. Using $\triangle ONP$ we obtain $(R - x)^2 + h^2 = R^2$ and using $\triangle QNP$ we obtain $(r - 2x)^2 + h^2 = r^2$. The two equations simplify to $x^2 - 2Rx + h^2 = 0$ (1) and $4x^2 - 4rx + h^2 = 0$ (2). Subtracting (1) from (2) and dividing by x yields the condition: $3x = 4r - 2R$. Since $OQ = 324$, then $OQ = R + r - 3x = 324 \to 3x = R + r - 324$. Thus, $R + r - 324 = 4r - 2R \to 3R - 3r = 324 \to R - r = \boxed{108}$.

7. For $0 < x < 1$, let $f(x) = (1 + x)(1 + x^4)(1 + x^{16})(1 + x^{64})(1 + x^{256}) \cdots$. Compute $f^{-1}\left(\frac{8}{5f(3/8)}\right)$ (2001 ARML I-8)
If $f(x) = (1 + x)(1 + x^4)(1 + x^{16})(1 + x^{64}) \cdots$, then $f(x^2) = (1 + x^2)(1 + x^8)(1 + x^{32})(1 + x^{128}) \cdots$ and $f(x)f(x^2) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16}) \cdots = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}$. Thus, $f(x^2) = \frac{1}{f(x)(1-x)}$. Letting $x = \frac{3}{8}$, we obtain $f\left(\frac{9}{64}\right) = \frac{1}{f\left(\frac{3}{8}\right)\left(1-\frac{3}{8}\right)} = \frac{8}{5f\left(\frac{3}{8}\right)}$. Thus, $\frac{9}{64} = f^{-1}\left(\frac{8}{5f(3/8)}\right)$. Answer: $\boxed{\frac{9}{64}}$.
More rigorously, let $g(x) = f(x)f(x^2) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \cdots$. We claim that $g(x) = \frac{1}{1-x}$ because $g(x)(1 - x) = (1 - x)\left((1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \cdots\right) = (1 - x^2)(1 + x^2)(1 + x^4)(1 + x^8) \cdots = (1 - x^4)(1 + x^4)(1 + x^8) \cdots = (1 - x^n)g(x^n)$. For $0 < x < 1$, each expression approaches 1 as n gets infinitely large, so $g(x)(1 - x) = 1$. Therefore, $f\left(\frac{3}{8}\right) f\left(\frac{9}{64}\right) = \frac{1}{1-(3/8)} = \frac{8}{5}$, so $f\left(\frac{9}{64}\right) = \frac{8}{5f(3/8)}$. Therefore, $f^{-1}\left(\frac{8}{5f(3/8)}\right) = \frac{9}{64}$.